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M a t h e m a t i c s

AUTOMORPHISMS OF FREE BURNSIDE GROUPS OF PERIOD 3

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We have proved that any automorphism of the free Burnside group $B(3)$ of period 3 and an arbitrary rank is induced by an automorphism of the free group of the same rank.

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Introduction. Let F be a free group and V a characteristic subgroup of F. Then the natural homomorphism from F to F/V gives rise to a homomorphism

$$\chi : \mathrm{Aut}(F) \to \mathrm{Aut}(F/V)$$

from the automorphism group of F to the automorphism group of F/V.

By definition the free Burnside group $B(X,n)$ of period n and basis X is the quotient group of the absolutely free group $F = F(X)$ with basis X by characteristic subgroup F^n generated by elements of the form a^n for all $a \in F(X)$.

T h e o r e m **.** Let $B(X,3) = F(X)/F(X)^3$ be a free Burnside group of period 3 with an arbitrary basis X of some rank. Then every automorphism of $B(X,3)$ is induced by an automorphism of the absolutely free group $F(X)$.

In the paper [1] we proved the theorem when X is finite. In the paper [2] Bryant and Macedonska proved that every automorphism of F/V is induced by an automorphism of F when F/V is nilpotent group of infinite rank. Bryant and Romankov proved even more general case in [3] when F/V is a free group of infinite rank in a subvariety of N_kA for some k. It is well known that a free Burnside group of period 3 is nilpotent, from which follows the truth of theorem when X is infinite.

In this paper we are going to give straight and short proof of the theorem when X is infinite using some results from the paper [2].

Bryant and Macedonska in [2] used so called *finitary lifting property*. Now we shall give the definition of the finitary lifting property. Let F be a free group of infinite rank and let $\{x_i : i \in I\}$ be a basis of F (for any relatively free group we use

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the term “basis” as an alternative to “free generating set”). An automorphism ξ of F will be called *finitary*, if there is a finite subset Ω of I such that $\xi(x_i) = x_i$ for all $i \in I \setminus \Omega$.

Let $\mathcal{B}$ be a variety of groups and write $V = \mathcal{B}(F)$. Suppose that Γ and Δ are subsets of I such that $\Gamma \cap \Delta$ is empty, Δ is finite, and $I \setminus (\Gamma \cup \Delta)$ is infinite. Let α be an automorphism of F/V such that $\alpha(x_iV) = x_iV$ for all $i \in \Gamma$. We say that the triple (Γ, Δ, α) can be lifted, if there exists a finitary automorphism ξ of F such that $\xi(x_i) = x_i$ for all $i \in \Gamma$ and $\xi(x_i)V = \alpha(x_iV)$ for all $i \in \Delta$. Such a finitary automorphism ξ is called a lifting of (Γ, Δ, α). We say that $\mathcal{B}$ has the finitary lifting property if, for every F of infinite rank every triple (Γ, Δ, α) can be lifted.

Proposition 1. [2]. Every nilpotent variety of groups has the finitary lifting property

Proposition 2. [2]. If $\mathcal{B}$ is any variety of groups with the finitary lifting property and F is a free group of infinite rank, then every automorphism of $F/\mathcal{B}(F)$ is induced by an automorphism of F.

Below we will give a direct proof that variety of free Burnside groups of period 3 has the finitary lifting property.

Let us recall the definitions of some automorphisms, which we will use later in the paper.

Let R be a relatively free group with the basis $X = \{x_i \in I\}$. Any homomorphism from R into itself is completely determined by the images of the basis elements. For any $x_i \in X$ let ε_i be the automorphism mapping x_i to x_i^{-1} and leaving other elements of X unchanged. For any different $x_i, x_j \in X$, let λ_{ij} be the automorphism mapping x_i to x_ix_j and leaving other elements of X unchanged. Automorphisms $\varepsilon_i, \lambda_{ij}$ are called Nielsen automorphisms. In 1924 Nielsen (see, for example, [4]) showed that the Nielsen automorphisms generate the full automorphism group $\mathrm{Aut}(F_n)$ of the finitely generated absolutely free group F_n.

Preliminary Lemmas. By $B(3)$ we denote a free Burnside group of period 3 with an arbitrary basis X of some rank. We need some commutator identities (Ch. 10, [5])

$$[a,b]^{-1} = [b,a], \tag{1}$$

$$[a,bc] = [a,c][a,b]^c. \tag{2}$$

Also we need some commutator identities that holds in free Burnside groups of period 3 and any rank (Ch. 5.12, [6]). For any generator $x_i \in X$ and for any element $g_i \in B(3)$ we have the equations:

$$[x_i, x_j, x_p] \neq 1 \text{ for different } i, j, p, \tag{3}$$

$$[g_1, g_2, g_3] = [g_3, g_1, g_2] = [g_2, g_3, g_1], \tag{4}$$

$$[g_1, g_2, g_3, g_4] = 1 \tag{5}$$

.

Lemma 1. (Ch. 18, [5]). For any element $u \in B(3)$ and for any generator $x_i \in X$ one of the following equalities:

$$u = u_1, \tag{6}$$

$$u = u_1 x_i u_2, \tag{7}$$

$$u = u_1 x_i^{-1} u_2, \tag{8}$$

$$u = u_1 x_i u_2 x_i^{-1} u_3 \tag{9}$$

holds for some $u_1, u_2, u_3 \in Gp(X \setminus x_i)$.

Lemma 2. An element g of the group $B(3)$ belongs to the commutator subgroup if and only if order of any generator in g by modulo 3 equals to 0.

Proof. The direct part of the claim is obvious. Let us show that if the order of any generator in g by modulo 3 equals to 0, then g belongs to the commutator subgroup. Let $g = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k}$, where $i_m \neq i_{m+1}$. Let's use induction with respect to the length of the word k. Not that if $g = x_{i_1}^{\varepsilon_1} U x_{i_1}^{\varepsilon} V$, then $g = x_{i_1}^{\varepsilon_1} U x_{i_1}^{-\varepsilon_1} U^{-1} U x_{i_1}^{\varepsilon_1+\varepsilon} V$. It is obvious that $x_{i_1}^{\varepsilon_1} U x_{i_1}^{-\varepsilon_1} U^{-1}$ belongs to the commutator subgroup. The element $U x_{i_1}^{\varepsilon_1+\varepsilon} V$ also belongs to the commutator subgroup, since it has the same generators' orders as g, but has a smaller length.

Lemma 3. For any automorphism $\alpha \in \mathrm{Aut}(B(3))$ the image of the generator x_i does not belong to the commutant of group $B(3)$.

Proof. Assume the converse, then we shall prove that the element $g = \alpha([x_i, x_j, x_p])$ is trivial, which contradicts to the definition of automorphism. Let $\alpha(x_i) = [g_{i_1}, g_{i_2}]...[g_{i_{2k-1}}, g_{i_{2k}}]$. The proof is by induction on k. In the case of $k = 1$ we have

$$g = [[g_{i_1}, g_{i_2}], \alpha(x_j), \alpha(x_p)] = [g_{i_1}, g_{i_2}, \alpha(x_j), \alpha(x_p)] = 1.$$

Suppose that the statement holds for $k-1$ and show it holds for k. From the Eqs. (2), (4) we get

$$g = [[g_{i_1}, g_{i_2}]...[g_{i_{2k-1}}, g_{i_{2k}}], \alpha(x_j), \alpha(x_p)] = [\alpha(x_j), \alpha(x_p), [g_{i_1}, g_{i_2}]...[g_{i_{2k-1}}, g_{i_{2k}}]] =$$
$$= [\alpha(x_j), \alpha(x_p), ([g_{i_3}, g_{i_4}]...[g_{i_{2k-1}}, g_{i_{2k}}])] \cdot [\alpha(x_j), \alpha(x_p), [g_{i_1}, g_{i_2}]]^{([g_{i_3}, g_{i_4}]...[g_{i_{2k-1}}, g_{i_{2k}}])}.$$

To prove $g = 1$ let us show that both multipliers are trivial. From the Eq. (4) it follows that $[\alpha(x_j), \alpha(x_p), ([g_{i_3}, g_{i_4}]...[g_{i_{2k-1}}, g_{i_{2k}}])] = [([g_{i_3}, g_{i_4}]...[g_{i_{2k-1}}, g_{i_{2k}}]), \alpha(x_j), \alpha(x_p)]$ and by inductive hypothesis the first multiplier is trivial. Again using the Eq. (4) we get $[\alpha(x_j), \alpha(x_p), [g_{i_1}, g_{i_2}]] = [[g_{i_1}, g_{i_2}], \alpha(x_j), \alpha(x_p)]$ from which follows the triviality of the second multiplier.

Theorem. Variety of groups $B(3)$ has the finitary lifting property.

Proof. Let (Δ, Γ, α) be an arbitrary triple, for which $\Gamma \cap \Delta$ is empty, Δ is finite and $I \setminus (\Gamma \cup \Delta)$ is infinite. Also $\alpha(x_i) = x_i$ for all $\imath \in \Gamma$ (x_i is the corresponding coset of the generator x_i, in this case, the coset $x_i B(3)$). Without loss of generality it can be assumed that $\Delta = \{x_1, \dots, x_k\}$. We say that the generator x_i is dominant in the element g, if the order of x_i in g by modulo 3 is not 0. For any automorphism $\alpha \in \mathrm{Aut}(B(3))$ and for any $\imath \in I$ there is a dominant in the image $\alpha(x_i)$.

Indeed, otherwise $\alpha(x_i)$ is in the commutant by Lemma 2, which is not possible by Lemma 3. Suppose $\{x_{j_1},\dots,x_{j_p}\}$ is the set of dominant elements of $\alpha(x_1)$ and $x_{j_i}\in\Gamma$, then from the properties of Γ it follows that $\alpha(x_{j_i})=x_{j_i}$ for $x_{j_i}\in\{x_{j_1},\dots,x_{j_p}\}$. Let us examine the automorphism $\alpha\lambda_{1j_1}{}^{\varepsilon_1}\dots\lambda_{1j_p}{}^{\varepsilon_p}$. With a proper selection of $\varepsilon_1,\dots\varepsilon_p$ we can exclude dominant elements in $\alpha\lambda_{1j_1}{}^{\varepsilon_1}\dots\lambda_{1j_p}{}^{\varepsilon_p}(x_1)$. Thus there is x_m in $\{x_{j_1},\dots,x_{j_p}\}$ such that $x_m\notin\Gamma$. From Lemma 1 $\alpha(x_1)=u_1x_m^{\varepsilon}v_1$, where $u_1,v_1\in Gp(X\setminus x_m)$. Using Nielsen's automorphisms and automorphisms P_{ij} (P_{ij} is the automorphism which permutes generators x_i,x_j leaving other elements of X unchanged) we will construct automorphism ξ_1 for which $\xi_1(x_1)=u_1x_m^{\varepsilon}v_1$. From the construction of the automorphism ξ_1 we see that $\xi_1(x_i)=x_i$ for $x_i\in\Gamma$ (since in the construction there were only used the automorphisms $\lambda_{mi},\varepsilon_m,P_{1m}$). For automorphism ξ_1^{-1} holds $\xi_1{}^{-1}(u_1x_m^{\varepsilon}v_1)=x_1$ identity. Let us examine the multiplication of automorphisms induced by the automorphism $\xi_1{}^{-1}$ and α. For the image of the generator x_1 we have $\xi^{-1}\alpha(x_1)=x_1$. Let $\Gamma_1=\Gamma\cup x_1$. Repeating the same argument for the generator x_2 and considering Γ_1 instead of Γ, using $\xi_1{}^{-1}\alpha$ instead of α we will get an automorphism ξ_2 so that $\xi_2(x_2)=\xi_1{}^{-1}\alpha(x_2)$ and $\xi_2(x_i)=x_i$ for $x_i\in\Gamma_1$. Thus $\xi_2{}^{-1}\xi_1{}^{-1}\alpha(x_1)=x_1,\xi_2{}^{-1}\xi_1{}^{-1}\alpha(x_2)=x_2$. Continuing this process till x_k, we will get automorphisms $\xi_1,\xi_2,\dots,\xi_k$, for which we have $\xi_k{}^{-1}\dots\xi_1{}^{-1}\alpha(x_i)=x_i$ for $x_i\in\Delta$, thus $\xi_1\dots\xi_k(x_i)=\alpha(x_i)$ for $x_i\in\Delta$. Since for any ξ_l we have $\xi_l(x_i)=x_i$ for $x_i\in\Gamma$, the same is true for $\xi_1\dots\xi_k$.

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