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Mathematics

DISCONTINUOUS RIEMANN BOUNDARY PROBLEM IN WEIGHTED SPACES

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The Riemann boundary problem in weighted spaces $L^1(\rho)$ on $T=\{t,|t|=1\}$, where $\rho(t)=|t-t_0|^\alpha$, $t_0\in T$ and $\alpha>-1$, is investigated. The problem is to find analytic functions $\Phi^+(z)$ and $\Phi^-(z)$, $\Phi^-(\infty)=0$ defined on the interior and exterior domains of T respectively, such that: $\lim\limits_{r\to 1-0}\|\Phi^+(rt)-a(t)\Phi^-(r^{-1}t)-f(t)\|_{L^1(\rho)}=0$, where $f\in L^1(\rho)$, $a(t)\in H_0(T;t_1,t_2,\dots,t_m)$. The article gives necessary and sufficient conditions for solvability of the problem and with explicit form of thr solutions.

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Introduction. Let $T=\{t,|t|=1\}$, $\rho(t)=|t-t_0|^\alpha$, where $t_0\in T$ and $\alpha>-1$ is an arbitrary real number. Let denote by $H(T)$ the Hölder class functions in T. We say that $a(t)\in H_0(T;t_1,t_2,\dots,t_m)$, if a belongs to the Hölder class on any interval from T not including $t_k, k=1,\dots,n$, points and has jump discontinuity at those points. Denote $D^+=\{z,|z|<1\}$, $D^-=\{z,|z|>1\}$, and let $D=D^+\cup D^-$. We shall consider the case $t_0\neq t_k$, $k=1,\dots,m$, and suppose that $a(t)\in H_0(T;t_1,t_2,\dots,t_m)$, $a(t)\neq 0$, $t\in T$. Introducing the function $\varphi(t)=\ln a(t)$, it is easy to get:

$$\alpha_k+i\beta_k=\frac{1}{2\pi i}(\varphi(t_k-0)-\varphi(t_k+0)),\ k=1,\dots,m.$$

Obviously the function $S_1(z)=exp\left\{\frac{1}{2\pi i}\int_T\frac{\varphi(t)dt}{t-z}\right\}$, $z\in D$, on a small neighborhood of any point t_k, $k=1,\dots,m$,can be represented by $S_1(z)=(z-t_k)^{\alpha_k+i\beta_k}\Delta_k(z)$, where $\Delta_k(z)$ is an analytic function on D and $\lim\limits_{z\to t_k}\Delta_k(z)=A\neq 0$. For the function $S_1(z)$ we will use following notation:

$$S_1(z)=\begin{cases}S_1^+(z), & z\in D^+,\\ \\ S_1^-(z), & z\in D^-.\end{cases}$$

Let consider the following boundary problem. That is, find analytic $\Phi(z)$ function on D such that

$$\Phi^+(t)-\Phi^-(t)=0, \tag{1}$$

where $$\Phi(z)=\begin{cases}\Phi^+(z), & z\in D^+,\\ \Phi^-(z), & z\in D^-.\end{cases}$$

There exist conformal mappings $\mu^+(z)$ and $\mu^-(z)$ $(\mu(\infty)=\infty)$ from D^+ and D^- into some domains Δ_+ and Δ_- respectively such that they satisfy Lipschitz condition in $D^+\cup T$ and $D^-\cup T$. Consider the functions

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$$\Phi_1^+(z)=\prod_{k=1}^{n}(\mu^+(z)-\mu^+(t_k))^{\lambda_k},\ z\in D^+;\ \ \Phi_1^-(z)=\prod_{k=1}^{n}(\mu^-(z)-\mu^-(t_k))^{\lambda_k},\ z\in D^-,$$

where λ_k are integers such that $-1<\lambda_k+\alpha_k\le 0,\ k=1,\dots,m.$

Denoting

$$\Phi_1(z)=\begin{cases}\Phi_1^+(z), & z\in D^+,\\[2mm] \Phi_1^-(z), & z\in D^-,\end{cases}$$

we can easily conclude that the function $\Phi_1(z)$ satisfies Eq. (1). Besides we have, $\Phi_1(z)=(z-t_k)^{\lambda_k}\Omega_k(z)$, where $\Omega_k(z)$ is analytic function in D and $\lim\limits_{z\to t_k}\Omega_k(z)=B\ne 0.$

Let $S(z)=S_1(z)\Phi_1(z),\quad z\in D^+\cup D^-.$

L e m m a . For the function $S(z)$ we have (see [1,2]):

a) $S^+(t)-a(t)S^-(t)=0$; b) $S^+(t),\ S^-(t)\in L^1(T)$ and $(S^+(t))^{-1},\ (S^-(t))^{-1}\in L^\infty(T)$;

c) on some neiborhood interval T_k of the point t_k the following is true:

$$S(z)=\frac{\lambda_k(z)}{(z-t_k)^{\delta_k-i\beta_k}},\ z\in D\cap T_k, \tag{2}$$

where $\delta_k=-(\alpha_k+\lambda_k)$, $0\le\delta_k<1$ and $\lambda_k(z)=\Delta_k(z)\Omega_k(z)$. Obviously $\lambda_k(z)\to AB\ne 0$, as $z\to t_k$. Let

$$n=\begin{cases}[\alpha]+1, & \text{if } \alpha \text{ is not integer},\\ \alpha, & \text{if } \alpha \text{ is integer.}\end{cases}$$

Problem R. Let $T=\{t,|t|=1\}$ be the unit circl, $a(t)\in H_0(T;t_1,t_2,\dots,t_m)$ and $a(t)\ne 0,\ t\in T$. $\rho(t)=|t-t_0|^\alpha$ is the weight function, where $\alpha>-1$ is an arbitrary real number and $t_0\in T$ such that $t_0\ne t_k,\ k=1,\dots,m$. Besides the function f belongs to the classes $L^1(\rho)$ in T. Find an analytic function $\Phi(z)$, $\Phi(\infty)=0$ on $D=D^+\cup D^-$, where $D^+=\{z,|z|<1\}$, $D^-=\{z,|z|>1\}$, satisfying the condition:

$$\lim_{r\to 1-0}\|\Phi^+(rt)-a(t)\Phi^-(r^{-1}t)-f(t)\|_{L^1(\rho)}=0. \tag{3}$$

Similarly, we will consider Problem **R** without the condition $\Phi(\infty)=0$. Instead we will suppose that the function Φ has some finite degree at infinity. Riemann boundary problem on weighted spaces for $a\in C^\delta$, $\delta\in(0,1]$, function was investigated in [3–5].

Main Results.

T h e o r e m 1. Let $f\in L^1(\rho)$. If Φ is a solution of the Problem **R** and has some finite degree at infinity, then the following representation is true:

$$\begin{aligned}\Phi^+(z)&=\frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T\frac{g(t)dt}{(t-z)},\ z\in D^+,\\ \Phi^-(z)&=\frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T\frac{g(t)dt}{(t-z)}+S^-(z)P(z),\ z\in D^-,\end{aligned} \tag{4}$$

where P is some polynomial and $g(t)=\left(P(t)+f(t)/S^+(t)\right)(t-t_0)^n$.

P r o o f . Let Φ be a solution of the Problem **R** and has finite degree at infinity. Let $f_r(t)\in H(T)$ be a sequence such that $\lim\limits_{r\to 1-0}\|f_r(t)-f(t)\|_{L^1(\rho)}=0$. From Lemma we can easily get the following result:

$$\lim_{r\to 1-0}\left\|\left(\frac{\Phi^+(rt)}{S^+(t)}-\frac{\Phi^-(r^{-1}t)}{S^-(t)}-\frac{f_r(t)}{S^+(t)}\right)(t-t_0)^n\right\|_{L_1}=0.$$

Let $\Psi_r(t)=\left(\dfrac{\Phi^+(rt)}{S^+(t)}-\dfrac{\Phi^-(r^{-1}t)}{S^-(t)}-\dfrac{f_r(t)}{S^+(t)}\right)(t-t_0)^n$ and

$$F_r^+(z)=\frac{\Phi^+(rz)(z-t_0)^n}{S^+(z)},\ z\in D^+,\ \ F_r^-(z)=\frac{\Phi^-(r^{-1}z)(z-t_0)^n}{S^-(z)},\ z\in D^-. \tag{5}$$

We have $F_r^+(t) - F_r^-(t) = \Psi_r(t) + \dfrac{f_r(t)(t-t_0)^n}{S^+(t)}$, where $t \in T$, $0 < r < 1$, $\Psi_r(t) \in H_0(T)$ and $\lim\limits_{r\to 1-0} \|\Psi_r(t)\|_{L^1} = 0$. Taking into account that the functions $F_r^+(z)$ and $F_r^-(z)$ are bounded on D^+ and D^- respectively, we conclude:

$$\begin{cases} F_r^+(z) = \dfrac{1}{2\pi i}\displaystyle\int_T \dfrac{g_r(t)dt}{t-z}, \\ F_r^-(z) = \dfrac{1}{2\pi i}\displaystyle\int_T \dfrac{g_r(t)dt}{t-z} + P_r(z), \end{cases} \tag{6}$$

where $P_r(z)$ is the principal part of the Laurent expansion of the function F_r^- at infinity and

$$g_r(t) = \Psi_r(t) + \left(P_r(t) + \frac{f_r(t)}{S^+(t)}\right)(t-t_0)^n.$$

We have $\lim\limits_{r\to 1-0} F_r^+(z) = \dfrac{\Phi^+(z)(z-t_0)^n}{S^+(z)}$, $\lim\limits_{r\to 1-0} F_r^-(z) = \dfrac{\Phi^-(z)(z-t_0)^n}{S^-(z)}$.
Besides, P_r uniformly converges to the polynomial P if $r \to 1-0$, as

$$\lim_{r\to 1-0} \| \left(\Psi(rt) + \frac{f_r(t)}{S^+(t)} - P(t) - \frac{f(t)}{S^+(t)}\right)(t-t_0)^n\|_{L_1} = 0,$$

we get $\lim\limits_{r\to 1-0} \|g_r(t) - g(t)\|_{L_1} = 0$. Taking into account (7), Theorem 1 is proved.

Theorem 2. Let $f \in L^1(\rho)$. Then the general solution of the Problem **R** with a finite degree at infinity is given by the following formula:

$$\begin{gathered} \Phi^+(z) = \frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T \frac{g(t)dt}{(t-z)},\ z \in D^+, \\ \Phi^-(z) = \frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T \frac{g(t)dt}{(t-z)} + S^-(z)P(z),\ z \in D^-, \end{gathered} \tag{7}$$

where P is any polynomial and $g(t) = \left(P(t) + \dfrac{f(t)}{S^+(t)}\right)(t-t_0)^n$.

Proof. We showed in Theorem 1, that any solution of the Problem **R** having finite degree at infinity has the representation (7). Now we will prove that any function, which has the representation (7) is a solution of the Problem **R**.

Let $f_n(t) \in H(T)$ and $\lim\limits_{n\to\infty} \|f_n(t) - f(t)\|_{L^1(\rho)} = 0$. Denote,

$$g_n(t) = \left(P(t) + \frac{f_n(t)}{S^+(t)}\right)(t-t_0)^n$$

Suppose

$$\begin{gathered} \Phi_n^+(z) = \frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T \frac{g_n(t)dt}{(t-z)},\ z \in D^+, \\ \Phi_n^-(z) = \frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T \frac{g_n(t)dt}{(t-z)} + S^-(z)P(z),\ z \in D^-. \end{gathered} \tag{8}$$

Since the functions $\Phi_n^+(z)$ and $\Phi_n^-(z)$ satisfy the Riemann boundary problem, we have $\Phi_n^+(t) - a(t)\Phi_n^-(t) = f_n(t)$ on T. Moreover, they have degree $-\delta$, $\delta \in (0,1)$ on some small neighborhood of the points t_k, so there exists a real number p, $p > 1$ such that for every r, $r \in (0,1)$ the following is true:

$$\int_T |\Phi_n^+(rt)|^p \rho(t)|dt| < C, \quad \int_T |\Phi_n^+(r^{-1}t)|^p \rho(t)|dt| < C.$$

Taking into account the last result, for every n, $n \geq 1$, we will have

$$\lim_{r\to 1-0} \|\Phi_n^+(rt) - a(t)\Phi_n^-(r^{-1}t) - f_n(t)\|_{L_1(\rho)} = \lim_{r\to 1-0} \|I_n^1(r)\|_{L_1(\rho)} = 0.$$

Denoting $\varepsilon_n(t)=g_n(t)-g(t)$, we will have

$$\int_T|\Phi^+(rt)-a(t)\Phi^-(r^{-1}t)-f(t)|\rho(t)|dt|\le$$

$$C\left(\int_T\frac{(1-r)|S^+(rt)|}{|t_0-rt|^n}\int_T\frac{|\varepsilon_n(\tau)|}{|\tau-rt|^2}|d\tau|\rho(t)|dt|+\int_T\frac{(1-r)|S^+(rt)|}{|t_0-rt|^{n+1}}\int_T\frac{|\varepsilon_n(\tau)|}{|\tau-r^{-1}t|}|d\tau|\rho(t)|dt|\right.$$

$$\left.+\int_T\frac{|S^+(rt)-a(t)S^-(r^{-1}t)|}{|t_0-r^{-1}t|^n}\int_T\frac{|\varepsilon_n(\tau)|}{|\tau-r^{-1}t|}|d\tau|\rho(t)|dt|\right)+\|I_n^1(r)\|_{L_1(\rho)}+\|f_n-f\|_{L_1(\rho)}.$$

All summands at the right side of the inequality tend to zero as $r\to 1-0$ [3,4]. □

Solution of the Problem. Let introduce the following functions: $\Phi_k^+(z)=\dfrac{1}{2\pi i}\displaystyle\int_T\frac{t^k(t-t_0)^n}{(t-z)}dt,\ z\in D^+,\quad \Phi_k^-(z)=\dfrac{1}{2\pi i}\displaystyle\int_T\frac{t^k(t-t_0)^n}{(t-z)}dt+z^k,\ z\in D^-$. Then, $\tilde g(t)=f(t)(t-t_0)^n(S^+(t))^{-1}$. Hence, for any polynomial $P(z)=c_0++c_1z+\dots+c_mz^m$ Eq. (7) can be represented as follows:

$$\begin{aligned}\Phi^+(z)&=\frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T\frac{\tilde g(t)dt}{(t-z)}+\frac{S^+(z)}{(z-t_0)^n}\sum_{k=0}^m c_k\Phi_k^+(z),\ z\in D^+,\\ \Phi^-(z)&=\frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T\frac{\tilde g(t)dt}{(t-z)}+\frac{S^-(z)}{(z-t_0)^n}\sum_{k=0}^m c_k\Phi_k^-(z),\ z\in D^-.\end{aligned}\tag{9}$$

Let $\kappa=-\sum_{k=1}^m\lambda_k$. Obviously the function S has κ degree at infinity. We say that κ is the index of function a.

Theorem 3. Let $f\in L^1(\rho)$. Then the general solution of the Problem **R** regarding to κ has the following representation:

a) if $n+\kappa\ge 0$, then

$$\begin{aligned}\Phi^+(z)&=\frac{S^+(z)}{2\pi i(z-t_0)^n}\int_T\frac{\tilde g(t)dt}{(t-z)}+\frac{S^+(z)}{(z-t_0)^n}\sum_{k=0}^{\kappa-1} c_k\Phi_k^+(z),\ z\in D^+,\\ \Phi^-(z)&=\frac{S^-(z)}{2\pi i(z-t_0)^n}\int_T\frac{\tilde g(t)dt}{(t-z)}+\frac{S^-(z)}{(z-t_0)^n}\sum_{k=0}^{\kappa-1} c_k\Phi_k^-(z),\ z\in D^-,\end{aligned}\tag{10}$$

where $c_0,c_1,\dots,c_{\kappa-1}$ are arbitrary complex numbers when $\kappa\ge 1$ and $c_0=c_1=\dots=c_{\kappa-1}$ when $\kappa=0$.

b) if $n+\kappa<0$, then the problem has a solution if and only if:

$$\int_T\frac{\tilde g(t)}{(t-z)}t^kdt=0,\ k=0,1,\dots,-(n+\kappa)-1.\tag{11}$$

Moreover, solution has the representation (11), where $c_0=c_1=\dots=c_{\kappa-1}$.

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